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New nonlinear coherent states associated with inverse bosonic and f-deformed ladder operators

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Abstract
Using the *nonlinear coherent states method*, a formalism for the construction of the coherent states associated with *'inverse bosonic operators'* and their dual family has been proposed. Generalizing the approach, the 'inverse of f-deformed ladder operators' corresponding to the nonlinear coherent states in the context of quantum optics and the associated coherent states have been introduced. Finally, after applying the proposal to a few known physical systems, particular nonclassical features as sub-Poissonian statistics and the squeezing of the quadratures of the radiation field corresponding to the introduced states have been investigated.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The *standard coherent states* $|z\rangle$ may be obtained from the action of a displacement operator on the vacuum [1],

$$D(z) = \exp(za^\dagger - z^* a), \qquad D(z)|0\rangle = |z\rangle, \tag{1}$$

or the right eigenstate annihilation operator

$$a|z\rangle = z|z\rangle, \tag{2}$$

where $z \in \mathbb{C}$ and $a, a^\dagger$ are the standard bosonic annihilation, creation operators, respectively. The states $|z\rangle$ are also minimum uncertainty states. It is well known that the expansion of these states in the Fock space is as follows:

$$|z\rangle = \exp^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{3}$$

where the set $\{|n\rangle\}_{n=0}^{\infty}$ is the number states of the quantized harmonic oscillator with Hamiltonian $\hat{H} = a^{\dagger}a + \frac{1}{2}$.

Due to the fact that the bosonic annihilation and creation operators, a and $a^{\dagger}$ are singular operators, the inverse operators a^{-1} and $a^{\dagger^{-1}}$ are not well defined. Nevertheless, the following generalized operators may be found in literature through the actions [2, 3]:

$$a^{\dagger^{-1}}|n\rangle = (1-\delta_{n,0})\frac{1}{\sqrt{n}}|n-1\rangle, \tag{4}$$

and

$$a^{-1}|n\rangle = \frac{1}{\sqrt{n+1}}|n+1\rangle, \tag{5}$$

where $\delta_{n,0}$ is 1 when $n = 0$, otherwise it is 0 (so by definition $a^{\dagger^{-1}}|0\rangle = 0$). Obviously, $a^{-1}\left(a^{\dagger^{-1}}\right)$ behaves like a creation (annihilation) operator. Also, the statement that a^{-1} $\left(a^{\dagger^{-1}}\right)$ is the right (left) inverse of a $(a^{\dagger})$ seems to be legally true, since

$$\begin{aligned} aa^{-1} &= a^{\dagger^{-1}}a^{\dagger} = \hat{I}, \\ a^{-1}a &= a^{\dagger}a^{\dagger^{-1}} = \hat{I} - |0\rangle\langle 0|, \end{aligned} \tag{6}$$

where $\hat{I}$ is the unit operator. Thus one may get the commutation relations $[a, a^{-1}] = |0\rangle\langle 0| = \left[a^{\dagger^{-1}}, a^{\dagger}\right]$. Anyway, the usefulness of these inverse operators in various contexts can be found in previous publications we shall address at this point. Indeed, the operators in (4) and (5) have been studied in para-Bose particles. In addition, these operators enable one to find the eigenvalue equation for the squeezed coherent states. The vacuum and the first exited squeezed states are the eigenstates of $a^{\dagger^{-1}}a$ and $aa^{\dagger^{-1}}$, respectively [4]. Therefore, for instance, since the squeezed vacuum states can be generated through some nonlinear optical processes, it is remarkable that inverse bosonic operators may play an important role in studying the time evolution of some nonlinear systems. Also, the important role of these operators has been followed in the metaplectic group structure of the $Mp(2)$ group, which is a two-fold cover of $Sp(2, R)$ and $SU(1, 1)$ groups [5]. Let us recall that *'photon added coherent states'* first introduced by Agarwal and Tara [6] are closely related to the inverse bosonic operators [3], where it has been shown that these states denoted by $|z, m\rangle = a^{\dagger^{m}}|z\rangle$ are eigenstates of the operator $a - ma^{\dagger^{-1}}$ with eigenvalues z. Subsequently, *'photon subtracted coherent states'* can be obtained by m-times actions of a^{-1} on $|z\rangle$, followed by m-times actions of $a^{\dagger^{-1}}$ on the resultant states, i.e. $|z, -m\rangle = a^{\dagger^{-m}}a^{-m}|z\rangle$.

Nowadays generalization of coherent states besides their experimental generations have made much interest in quantum physics, especially in quantum optics [1]. These quantum states exhibit some interesting *'non-classical properties'* particularly squeezing, antibunching, sub-Poissonian statistics and oscillatory number distribution. Along achieving this goal, the first purpose of the present paper is to outline a formalism for the construction of coherent states associated with the *'inverse bosonic operators'*, the states that have not been found in the literature up to know. But before paying attention to this matter, a question may naturally arise about the relation between the operators $a^{\dagger^{-1}}$ and a^{-1} and the standard coherent states $|z\rangle$. Due to the second equation in (6) it is readily found that the following does not hold: $a^{-1}|z\rangle = z^{-1}|z\rangle$, which at first glance may be expected from equation (2). Instead, one has

$$a^{-1}|z\rangle = z^{-1}[|z\rangle - e^{-\frac{1}{2}|z|^2}|0\rangle]. \tag{7}$$

This is consistent with the fact that the right eigenstate of the operator a^{-1} does not exist, originates from the creation-like characteristic of a^{-1} [2]. Also it can be seen that the standard coherent state $|z\rangle$ in (3) is not the eigenstate of $a^{\dagger^{-1}}$, one has instead

$$a^{\dagger^{-1}}|z\rangle = z\,\mathrm{e}^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}(n+1)}|n\rangle. \tag{8}$$

Nevertheless, as we will observe, the 'nonlinear coherent state method' provides a rich enough mathematical structure allowing us to establish our aim. We will illustrate that although the presented formalism yields a non-normalizable coherent state corresponding to the inverse bosonic operator $\left(a^{\dagger^{-1}}\right)$, the associated dual family is well defined. In the continuation of the paper, along generalization of the proposal to the *'inverse nonlinear (f-deformed) ladder operators'* involved in the nonlinear coherent states context in quantum optics, the associated generalized coherent states have been also introduced. Then, as some physical realizations of the proposed formalism, hydrogen-like spectrum, harmonious states and Gilmore–Perelomov representation of $SU(1,1)$ group have been considered. Taking into account their nonlinearity functions, we shall deduce the explicit form of the corresponding coherent states associated with the inverse f-deformed operators. At last, we conclude the paper with investigating some interesting nonclassical properties, for instance sub-Poissonian statistics (anti-bunching) and the squeezing of the quadratures of the field of the obtained states, numerically.

2. Coherent states of inverse bosonic operators

In this section after presenting a brief review of the nonlinear coherent states, we are going to establish a link between the 'inverse bosonic' and 'f-deformed (nonlinear) ladder' operators.

2.1. The link between 'inverse bosonic' and 'f-deformed ladder' operators

The notion of *'nonlinear'* or '*f-deformed*' coherent states which provides a powerful method to analyze a large number of the quantum optics states [7–9]. Any class of these states, characterized by a particular intensity-dependent function $f(n)$ is defined as the solution of the typical eigenvalue equation $a_f|z,f\rangle = z|z,f\rangle$, with decomposition in the number states space as

$$|z,f\rangle = N(|z|^2)^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}[f(n)]!}|n\rangle, \tag{9}$$

where $a_f = af(n)$ is the f-deformed annihilation operator, $[f(n)]! \doteq f(n)f(n-1)f(n-2)\cdots f(2)f(1)$ and $[f(0)]! \doteq 1$. The function $N(|z|^2)$ in (9) is the normalization constant which can readily be calculated as $\sum_{n=0}^{\infty}|z|^{2n}/[nf^2(n)]!$. Choosing different $f(n)$'s lead to distinct generalized coherent states.

The nonorthogonality (as a consequence of overcompleteness) of the states in (9), i.e. $\langle z,f|z',f\rangle \neq 0$ (and all the new states will be introduced in the present paper) is so clear that we pay no attention to it. These states are required to satisfy the resolution of the identity

$$\int_D \mathrm{d}^2z|z,f\rangle W(|z|^2)\langle z,f| = \sum_{n=0}^{\infty}|n\rangle\langle n| = \hat{I}, \tag{10}$$

where $\mathrm{d}^2z \doteq \mathrm{d}x\,\mathrm{d}y$, $W(|z|^2)$ is a positive weight function may be found after specifying $f(n)$, and D is the domain of the states in the complex plane defined by the disk

$$D = \{z \in \mathbb{C}, |z| \leqslant \lim_{n\to\infty}[nf^2(n)]\}, \tag{11}$$

centered at the origin in the complex plane. Inserting the explicit form of the states (9) in (10) with $|z|^2 \equiv x$ it can be easily checked that the resolution of the identity holds if the following moment problem is satisfied:

$$\pi \int_0^R \mathrm{d}x\, \sigma(x) x^n = [nf^2(n)]!, \qquad n = 0, 1, 2, \ldots, \tag{12}$$

where $\sigma(x) = \frac{W(x)}{N(x)}$ and R is the radius of convergence determined by the relation (11). Condition (12) presents a severe restriction on the choice of $f(n)$. Altogether, there are cases for which the completeness of some previously introduced coherent states has been demonstrated a few years later elsewhere (photon added coherent states were introduced in 1991 [6] while their completeness condition was demonstrated in 2001 [10]). In fact, only a relatively small number of $f(n)$ functions are known, for which the functions $\sigma(x)$ can be extracted.

The action of an f-deformed creation operator defined as $a_f^\dagger = f^\dagger(\hat{n})a^\dagger$ on the number states expresses as follows:

$$a_f^\dagger|n\rangle = f^\dagger(n+1)\sqrt{n+1}|n+1\rangle. \tag{13}$$

Now, going back to our goal in the paper, comparing equations (13) and (5) it can be easily seen that,

$$a^{-1} \equiv a_f^\dagger = f^\dagger(\hat{n})a^\dagger, \qquad \text{with} \quad f(\hat{n}) = \frac{1}{\hat{n}}. \tag{14}$$

Similarly, using the action of the f-deformed annihilation operator a_f on the number states, i.e.,

$$a_f|n\rangle = f(n)\sqrt{n}|n-1\rangle \tag{15}$$

and then comparing with (4) one readily finds

$$a^{\dagger^{-1}} \equiv a_f = af(\hat{n}), \tag{16}$$

with the same $f(\hat{n})$ introduced in (14). Note that the following also holds

$$a^\dagger f^\dagger(\hat{n}) = f^\dagger(\hat{n}-1)a^\dagger, \qquad f(\hat{n})a = af(\hat{n}-1). \tag{17}$$

Therefore, taking into account all the above results we can write the explicit forms of the inverse bosonic operators denoted by a_f and $a_f^\dagger$, and the related actions as follows [7]:

$$a_f \equiv a^{\dagger^{-1}} = a\frac{1}{\hat{n}}, \qquad a_f|n\rangle = (1-\delta_{n,0})\frac{1}{\sqrt{n}}|n-1\rangle, \tag{18}$$

$$a_f^\dagger \equiv a^{-1} = \frac{1}{\hat{n}}a^\dagger, \qquad a_f^\dagger|n\rangle = \frac{1}{\sqrt{n+1}}|n+1\rangle. \tag{19}$$

Actually in the latter equations the nonlinearity function is considered as $f(\hat{n}) = \frac{1}{\hat{n}}$. Equations (18) and (19) confirm that a_f $\left(a_f^\dagger\right)$ annihilates (creates) one (deformed) quanta of photon in some optical processes, respectively. For the commutation relation between the two ladder operators introduced into (18) and (19) one arrives at

$$\left[a_f, a_f^\dagger\right] = -\frac{1}{\hat{n}(\hat{n}+1)}, \qquad \text{for} \quad n \neq 0 \tag{20}$$

and

$$\left[a_f, a^\dagger{}_f\right] = |0\rangle\langle 0|, \qquad \text{for} \quad n = 0. \tag{21}$$

Interestingly, this method with the factorized Hamiltonian formalism, permits one to derive a Hamiltonian responsible to the dynamics of the (inverse) system as [9]

$$\hat{h} = a_f^\dagger a_f = \frac{1}{\hat{n}} \equiv \hat{H}^{-1} \qquad \text{for} \quad n \neq 0, \tag{22}$$

where one may define $\hat{h} \equiv \hat{H}^{-1} = 0$ for $n = 0$, consistent with the definitions in (4) and (5). The Hamiltonian in this case is the inverse of the Hamiltonian of the standard (shifted) harmonic oscillator $a^\dagger a$. Unlike the quantized harmonic oscillator, the spectrum of the new Hamiltonian system, $\hat{h}$, is not equally distanced (arises from the nonlinearity nature of the inverse system).

2.2. Introducing $|z, f\rangle^{(-1)}$ as the coherent states associated with $a^{\dagger^{-1}}$

Now, one may look for the right eigenstate of the annihilation-like operator a_f such that

$$a_f|z, f\rangle^{(-1)} = z|z, f\rangle^{(-1)}. \tag{23}$$

The superscript (-1) on any state $|\cdot\rangle$ (in the whole of the present paper) refers to the state corresponds to an 'inverse' operator. A straightforward calculation shows that the state $|z, f\rangle^{(-1)}$ satisfies the eigenvalue equation (23) and has the following expansion in the Fock space:

$$|z, f\rangle^{(-1)} = N(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \sqrt{n!} z^n |n\rangle, \tag{24}$$

with the normalization constant

$$N(|z|^2) = \sum_{n=0}^{\infty} n!|z|^{2n} \tag{25}$$

which clearly diverges. Therefore, precisely speaking the eigenstate of the annihilation-like operator $a_f \equiv a^{\dagger^{-1}}$ does really exist but unfortunately it is physically meaningless (due to non-normalizablity of the state). This is an expected result since the relation (11) determines the radius of convergence equal to 0 when $f(n) = \frac{1}{n}$, i.e. for the case in hand. So, the states in (24) cannot actually belong to the Hilbert space.

2.3. Introducing $|\tilde{z}, f\rangle^{(-1)}$ as the dual family of $|z, f\rangle^{(-1)}$

In what follows we will observe that the *dual family* of the states in (24) is well defined. For this purpose, it is possible to define two new operators b_f and $b_f^\dagger$ as follows:

$$b_f = a\frac{1}{f^\dagger(\hat{n})} = a\hat{n}, \qquad b_f^\dagger = \frac{1}{f(\hat{n})}a^\dagger = \hat{n}a^\dagger. \tag{26}$$

Thus, one has $\left[a_f, b_f^\dagger\right] = \hat{I} = \left[b_f, a_f^\dagger\right]$. These properties allow one to define the generalized (non-unitary) displacement operator as follows:

$$D_f(z) = \exp\left[z b_f^\dagger - z^* a_f\right], \tag{27}$$

the action of which on the vacuum of the field gives the already obtained state in (24). But, according to the proposal has been recently introduced in [11, 12] another displacement operator may also be constructed as

$$\widetilde{D}_f(z) = \exp\left[z a_f^\dagger - z^* b_f\right], \tag{28}$$

the action of which on the vacuum of the field gives a new set of nonlinear coherent states as

$$|\widetilde{z}, f\rangle^{(-1)} = \widetilde{D}_f(z)|0\rangle = \widetilde{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{3/2}} |n\rangle, \tag{29}$$

where $z \in \mathbb{C}$. The normalization constant $\widetilde{\mathcal{N}}(|z|^2)$ can be obtained as

$$\widetilde{N}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(n!)^3} = {}_0F_2(1, 1, |z|^2), \tag{30}$$

where ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_p,;z) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{z^k}{k!}$ is the generalized hypergeometric function and $(a)_m = \Gamma(a+m)/\Gamma(m)$ with $\Gamma(m)$ the well-known Gamma function. It can be observed that these states can be defined on the whole space of complex plane. Nowadays, the states in (9) and (29) are known under the name *'dual family'* or *'dual pair'* coherent states [11, 12]. It can be checked straightforwardly that the nonlinear coherent states in (29) are also the right eigenstates of the deformed annihilation operator $b_f = a\hat{n}$. Thanks to J R Klauder *et al* for they established the resolution of the identity of the states in (29) via the moment problem technique [13].

3. The inverse of the deformed annihilation (and creation) operator and the associated nonlinear coherent states

Generalizing the proposed approach to the F-deformed rising and lowering operators

$$A = aF(\hat{n}), \qquad A^{\dagger} = F^{\dagger}(\hat{n})a^{\dagger}, \tag{31}$$

corresponding to nonlinear oscillator algebra, one can define

$$A^{-1} = F^{-1}(\hat{n})a^{-1} = \frac{1}{\hat{n}F(\hat{n})}a^{\dagger} \equiv \mathcal{F}^{\dagger}(\hat{n})a^{\dagger} \tag{32}$$

and

$$A^{\dagger^{-1}} = a^{\dagger^{-1}}F^{\dagger^{-1}}(\hat{n}) = a\frac{1}{\hat{n}F^{\dagger}(\hat{n})} \equiv a\mathcal{F}(\hat{n}), \tag{33}$$

where in the third steps of the derivation of equations (32) and (33) the left equations of (19) and (18) have been used, respectively. In the continuation of the paper we shall call $F(n)$ the *'original'* nonlinearity function. It is worth mentioning two points. First is that the 'generalized nonlinearity function' $\mathcal{F}(\hat{n})$ has been defined in terms of the original nonlinearity function $F(n)$ as

$$\mathcal{F}(\hat{n}) = \frac{1}{\hat{n}F^{\dagger}(\hat{n})}, \tag{34}$$

and second since the original nonlinearity function $F(\hat{n})$ is considered to be an operator-valued function which generally can be complex [14], so is $\mathcal{F}(\hat{n})$. The number states representations of the operators in (32) and (33) take the forms

$$A^{-1} \doteq \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}F(n+1)}|n+1\rangle\langle n| \tag{35}$$

and

$$A^{\dagger^{-1}} \doteq \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}F(n+1)}|n\rangle\langle n+1|. \tag{36}$$

It can be seen that

$$\begin{aligned} AA^{-1} &= A^{\dagger^{-1}}A^{\dagger} = \hat{I}, \\ A^{-1}A &= A^{\dagger}A^{\dagger^{-1}} = \hat{I} - |0\rangle\langle 0|, \end{aligned} \tag{37}$$

which mean that A^{-1} is the right inverse of A, and $A^{\dagger^{-1}}$ is the left inverse of $A^{\dagger}$, analogously to the interpretation of the inverse bosonic operators. With the help of the action of operators in (35) and (36) on the number states one has

$$A^{\dagger^{-1}}|n\rangle = \frac{1}{\sqrt{n}F(n)}(1-\delta_{n,0})|n-1\rangle \tag{38}$$

and

$$A^{-1}|n\rangle = \frac{1}{\sqrt{n+1}F(n+1)}|n+1\rangle, \tag{39}$$

where by definition $A^{\dagger^{-1}}|0\rangle = 0$. Therefore, $A^{\dagger^{-1}}$ and A^{-1} act on the number states like annihilation and creation operators, respectively. We will rename reasonably thus the generalized inverse operators in (32) and (33) as

$$\mathcal{A}^\dagger \equiv A^{-1} = \mathcal{F}^\dagger(\hat{n})a^\dagger, \qquad \mathcal{A} \equiv A^{\dagger^{-1}} = a\mathcal{F}(\hat{n}), \tag{40}$$

respectively. Note that the following commutation relation holds

$$[\mathcal{A}, \mathcal{A}^\dagger] = (\hat{n}+1)|\mathcal{F}(\hat{n}+1)|^2 - \hat{n}|\mathcal{F}(\hat{n})|^2 \tag{41}$$

which can be expressed in terms of the F-function as

$$[\mathcal{A}, \mathcal{A}^\dagger] = \frac{1}{(\hat{n}+1)|F(\hat{n}+1)|^2} - \frac{1}{\hat{n}|F(\hat{n})|^2}, \qquad \text{for} \quad n \neq 0, \tag{42}$$

and

$$[\mathcal{A}, \mathcal{A}^\dagger] = |0\rangle\langle 0|, \qquad \text{for} \quad n = 0. \tag{43}$$

The dynamics of the 'inverse nonlinear oscillator' may be described by the (inverse) Hamiltonian

$$\hat{\mathcal{H}} = \mathcal{A}^\dagger\mathcal{A} = \frac{1}{\hat{n}|F(\hat{n})|^2} \equiv \hat{H}^{-1}, \qquad \text{for} \quad n \neq 0, \tag{44}$$

and $\hat{\mathcal{H}} = 0$ for $n = 0$. Interestingly, the Hamiltonian $\hat{\mathcal{H}}$ in (22) is the inverse of the Hamiltonian of the 'original nonlinear oscillator' which is a familiar feature in the nonlinear coherent states context.

Now, the corresponding F-coherent states using the algebraic definition

$$\mathcal{A}|z, \mathcal{F}\rangle^{(-1)} = z|z, \mathcal{F}\rangle^{(-1)} \tag{45}$$

may be demanded. A straightforward calculation shows that the states $|z, \mathcal{F}\rangle^{(-1)}$ have the following expansions:

$$|z, \mathcal{F}\rangle^{(-1)} = \mathcal{N}(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}[\mathcal{F}(n)]!}|n\rangle. \tag{46}$$

The states in (46) when transformed in terms of the original nonlinearity function we started with $(F(n))$, take the following form:

$$|z, F\rangle^{(-1)} = \mathcal{N}(|z|^2)^{-1/2}\sum_{n=0}^{\infty} z^n\sqrt{n!}[F^\dagger(n)]!|n\rangle, \tag{47}$$

where the definition $[\mathcal{F}(n)]! \doteq \frac{1}{n![F^\dagger(n)]!}$ has been used and

$$\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty} n!|z|^{2n}([F^\dagger(n)]!)^2. \tag{48}$$

With the particular choice of $F(n) = \frac{1}{n}$ in (47) (or equivalently $\mathcal{F}(n) = 1$ in (46)) the standard coherent state in (3), known as *self-dual* states, will be reobtained.

Similar to the procedure which led us to equation (12), the resolution of the identity requirement associated with the state in (46) (or (47)) has been satisfied if a function $\eta(x)$ is found such that

$$\pi \int_0^{\mathcal{R}} \mathrm{d}x\, \eta(x) x^n = [n\mathcal{F}^{\dagger 2}(n)]!, \\ = \left[\frac{1}{nF^{\dagger 2}(n)}\right]!, \qquad n = 0, 1, 2, \ldots, \tag{49}$$

where $\mathcal{R}$ is the radius of convergence determined by the disk

$$\mathcal{D} = \left\{z \in \mathbb{C}, |z| \leqslant \lim_{n\to\infty} \left[nF^{\dagger 2}(n)\right]^{-1}\right\}, \tag{50}$$

centered at the origin in the complex plane.

Related to the operators $\mathcal{A}^\dagger$ and its conjugate $\mathcal{A}$, two conjugate operators can be defined as

$$\mathcal{B} = a\frac{1}{\mathcal{F}^\dagger(n)}, \qquad \mathcal{B}^\dagger = \frac{1}{\mathcal{F}(n)} a^\dagger, \tag{51}$$

such that the following canonical commutation relations hold:

$$[\mathcal{A}, \mathcal{B}^\dagger] = \hat{I} = [\mathcal{B}, \mathcal{A}^\dagger]. \tag{52}$$

The relations in (51) and (52) enable one to define two generalized (non-unitary) displacement-type operators

$$D_{\mathcal{F}}(z) = \exp(z\mathcal{B}^\dagger - z^*\mathcal{A}) \tag{53}$$

and

$$\widetilde{D}_{\mathcal{F}}(z) = \exp(z\mathcal{A}^\dagger - z^*\mathcal{B}). \tag{54}$$

By the action of $D_{\mathcal{F}}(z)$ defined in (53) on the fundamental state one readily finds that

$$D_{\mathcal{F}}(z)|0\rangle = |z, \mathcal{F}\rangle^{(-1)}, \tag{55}$$

which are exactly the states obtained in equations (46) and (47) in terms of $\mathcal{F}(n)$ and $F(n)$, respectively. To this end, by the action of $\widetilde{D}_{\mathcal{F}}(z)$ in (54) on the vacuum one gets a new set of states

$$\widetilde{D}_{\mathcal{F}}(z)|0\rangle = |\widetilde{z}, \mathcal{F}\rangle^{(-1)} \\ = \widetilde{\mathcal{N}}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} [\mathcal{F}^\dagger(n)]!|n\rangle. \tag{56}$$

The latter states can be expressed in terms of the original function $F(n)$ as follows:

$$|\widetilde{z}, F\rangle^{(-1)} = \widetilde{\mathcal{N}}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}[nF^\dagger(n)]!}|n\rangle, \tag{57}$$

where

$$\widetilde{\mathcal{N}}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(n!)^3([F^\dagger(n)]!)^2}. \tag{58}$$

The resolution of the identity for the dual state in (56) (or (57)) has been satisfied if a positive function $\widetilde{\eta}(x)$ is found such that

$$\pi \int_0^{\widetilde{\mathcal{R}}} \mathrm{d}x\, \widetilde{\eta}(x) x^n = \left[n\frac{1}{\mathcal{F}^{\dagger 2}(n)}\right]!, \\ = \left[n^3 F^{\dagger 2}(n)\right]!, \qquad n = 0, 1, 2, \ldots, \tag{59}$$

where $\widetilde{\mathcal{R}}$ is the radius of convergence determined by the disk

$$\widetilde{\mathcal{D}} = \left\{z \in \mathbb{C}, |z| \leqslant \lim_{n\to\infty} \left[n^3 F^{\dagger 2}(n)\right]\right\}, \tag{60}$$

centered at the origin in the complex plane.

Upon substituting $F(n) = 1$ into (47) and (57) the states in (24) and (29) will be reobtained, respectively, i.e. the dual family of coherent states associated with the inverse of bosonic operator.

The states introduced into (46) and (56) (or equivalently in (47) and (57)) are the dual pair (nonlinear) coherent states corresponding to the generalized inverses of the deformed operators [11, 14]. Comparing the state in (57) and the usual nonlinear coherent state in (9) shows that a multiplication factor $n!$ appears in the denominator of the expansion coefficient of the usual nonlinear coherent state. Note that the existence of the factor $[F^{\dagger}(n)]!$ in the expansion coefficients of (47) and (48) (or (57) and (58)) provides a good potentiality which allows one to use suitable nonlinearity functions $F(n)$ for constructing a wide variety of well-defined generalized coherent states associated with inverse F-deformed operators.

4. Some physical realizations of the formalism and their nonclassical properties

Generally, a state is known as a nonclassical state (with no classical analogue) if the Glauber–Sudarshan $P(\alpha)$ function [15, 16] cannot be interpreted as a probability density. However, in practice one cannot directly apply this criterion to investigate the nonclassicality nature of a state [17]. So, this purpose has been frequently achieved by verifying *'squeezing, antibunching, sub-Poissonian statistics and oscillatory number distribution'*. A common feature of all the above-mentioned criteria is that the corresponding P-function of a nonclassical state is not positive definite. Therefore, each of the above effects (squeezing or sub-Poissonian statistics which we will consider in the paper) is indeed sufficient for a state to possess nonclassicality signature.

- *Sub-Poissonian statistics.* To examine the statistics of the states the Mandel's Q-parameter is used, which characterizes the quantum states of light in the cavity. Mandel's Q-parameter has been defined as

$$Q = \frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle} - 1. \tag{61}$$

  This quantity vanishes for *'standard coherent states'* (Poissonian), is positive for *'classical'* (bunching effect) and negative for *'nonclassical'* light (antibunching effect).
- *Squeezing phenomena*. Based on the following definitions of position and momentum operators:

$$x = \frac{a + a^{\dagger}}{\sqrt{2}}, \qquad p = \frac{a - a^{\dagger}}{\sqrt{2}\mathrm{i}}, \tag{62}$$

  the corresponding uncertainties will be defined as follows:

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2, \qquad (\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2. \tag{63}$$

  A state is squeezed in position or momentum quadrature if the uncertainty in the corresponding quadrature falls below the one's for the vacuum of the field, i.e. $(\Delta x)^2 < 0.5$ or $(\Delta p)^2 < 0.5$, respectively.

To give some physical realizations of the proposal, firstly one must specify the system with known '*nonlinearity function*' or '*discrete spectrum*' (these two quantities are related to each other through the relation $e_n = nf^2(n)$, where e_n denotes the spectrum of physical system [9, 14]). At this stage in this section we shall concern ourselves with three particular systems: 'hydrogen-like spectrum', 'harmonious state' and 'Gilmore–Perelomov representation of $SU(1,1)$ group', for all of which the corresponding usual nonlinear coherent states and nonlinearity natures have been previously clarified. The squeezing effect and Mandel's Q-parameter for the obtained states in the paper may be evaluated numerically. For this purpose one must calculate the expectation values expressed in (61) and (63) over any state of interest.

4.1. Hydrogen-like spectrum

As an important physical system we will accomplish in the present paper we want to apply our proposal onto the hydrogen-like spectrum. This quantum system is described by discrete spectrum,

$$e_n = 1 - \frac{1}{(n+1)^2}. \tag{64}$$

The nonlinearity function in this case has been expressed as [9, 14]

$$F_H(n) = \frac{\sqrt{n+2}}{n+1}. \tag{65}$$

The standard nonlinear coherent state corresponding to this nonlinear function obtained with the help of (9) is restricted to a unit disk in the complex plane centered at the origin. In this subsection, the nonclassicality nature of the dual pair of coherent states (according to the structural equations (47) and (57)) associated with hydrogen-like spectrum has been investigated. For the coherent states according to (47) in this example, the domain is restricted to $|z| < 1$, while the domain would be $z \in \mathbb{C}$ when the states are constructed from (57). The latter results are consistent with the general feature that occurs in the framework of the dual pair of coherent states, where if the domain of one set of a dual pair of coherent states is the whole of the complex plane, that of the other set would be the unit disk and vice versa [9]. Anyway, for instance, to verify the resolution of the identity for the corresponding dual states according to (57) we use the definition of Meijer's G-function together with the inverse Mellin theorem [18],

$$\int_0^\infty \mathrm{d}x\, x^{s-1} G^{m,n}_{p,q}\left(\alpha x \middle| \begin{matrix} a_1, & \cdots, & a_n, & a_{n+1}, & \cdots, & a_p \\ b_1, & \cdots, & b_m, & b_{m+1}, & \cdots, & b_q \end{matrix}\right) = \frac{1}{\alpha^s} \times \frac{\Pi_{j=1}^m \Gamma(b_j+s)\Pi_{j=1}^n \Gamma(1-a_j-s)}{\Pi_{j=n+1}^p \Gamma(a_j+s)\Pi_{j=m+1}^q \Gamma(1-b_j-s)}. \tag{66}$$

So, for instance, the function $\widetilde{\eta}(x)$ satisfying equation (59) with nonlinearity function introduced into (65) may be given in terms of the Meijer's G-function by the expression

$$\widetilde{\eta}(x) = G^{4,0}_{2,0}\left(x \middle| \begin{matrix} 0, 0, 0, 2 & . \\ 0, 0 & . \end{matrix}\right). \tag{67}$$

Thus, the associated weight function which satisfies the resolution of the identity for these set of states can be calculated as $\widetilde{\mathcal{W}}(x) = \widetilde{\eta}(x)(1+x)\,{}_0F_1(3,x)$, where ${}_0F_1$ is the regularized confluent hypergeometric function and $\widetilde{\eta}(x)$ is determined in (67). Now, the numerical results for the dual pair of coherent states according to (47) and (57) with the nonlinearity function in (65) will be displayed in figures 1–4. Although the dual pair of coherent states are defined

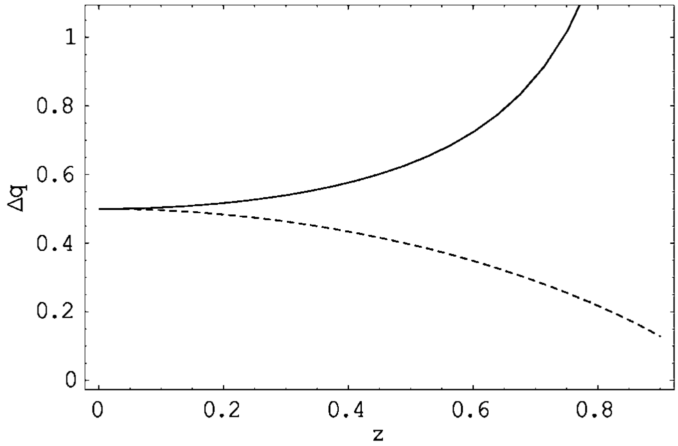

Figure 1. The uncertainties in x (the solid curve) and p (the dashed curve) as a function of $z \in \mathbb{R}$ for the hydrogen-like spectrum according to equation (47).

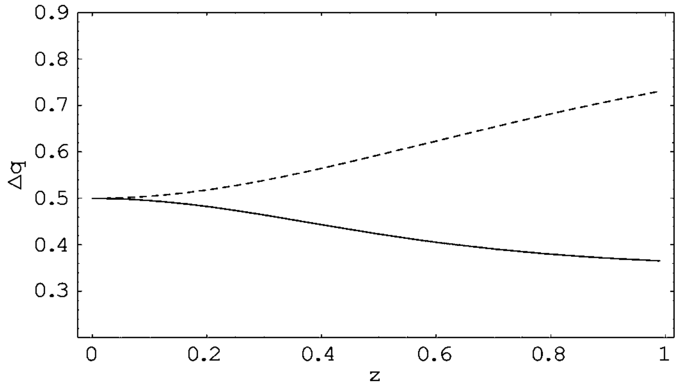

Figure 2. The same as figure 1 except that equation (57) has been considered.

on different domains, for the sake of comparison our numerical calculations for both of them presented just for $|z| < 1$. In figure 1, the uncertainties in x and p have been plotted in the respected domain as a function of $z \in \mathbb{R}$ utilizing (47). The squeezing in p-quadrature has been shown for real $z < 1$. Figure 2 is the same as figure 1 except that equation (57) has been considered. The squeezing in x-quadrature has occurred for real z. Our further computations when $z \gg 1$ for the example in hand upon using (57), as defined on the whole complex plane, indicated that the variances in x (solid line) and p (dashed line) tend respectively at about $\cong 0.25$ and $\cong 1$. So, the squeezing in x is visible for any real value of z. The three-dimensional graph of Mandel's Q-parameter as a function of $z \in \mathbb{C}$ has been shown in figure 3 for the states constructed according to (47). As it can be seen the sub-Poissonian statistics is restricted to a finite range of values of z near $z < 1$. Figure 4 is the same as figure 3 when (57) is used. In this case, the sub-Poissonian exhibition has occurred for all values of $z \in \mathbb{C}$. So, in view of this result, the latter are fully nonclassical states, in the sense that they have a nonclassical nature within the whole permitted range of z values.

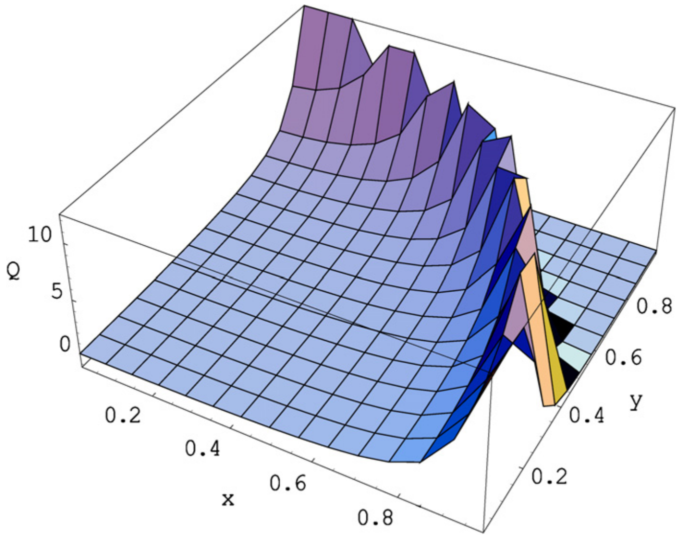

Figure 3. The graph of Mandel's Q-parameter as a function of $z \in \mathbb{C}$, for the hydrogen-like spectrum according to equation (47).

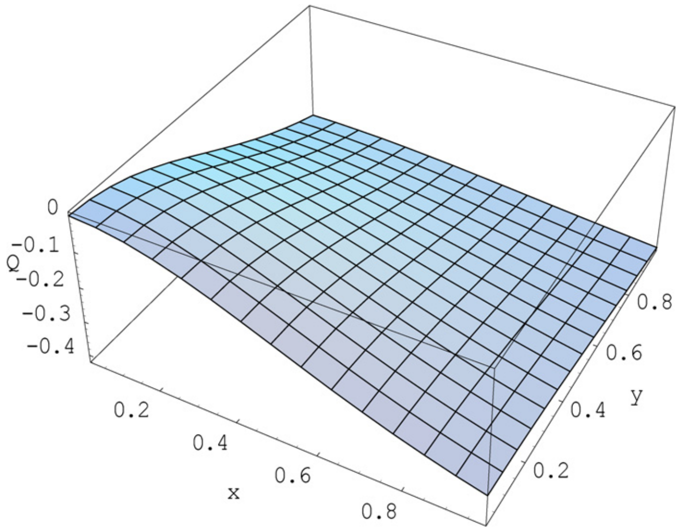

Figure 4. The same as figure 3 except that equation (57) has been considered.

4.2. Harmonious states

Harmonious states characterized by the nonlinearity function

$$F_{\mathrm{HS}}(n) = \frac{1}{\sqrt{n}} \tag{68}$$

has considerable attention in quantum optics. It can be observed that the lowering operator constructed from the nonlinearity function in (68) is equivalently the nonunitary Susskind–Glogower operator $\exp(\mathrm{i}\hat{\Phi}) = a(a^{\dagger}a)^{-1/2}$ [19]. It has been shown that the probability operator measures generated by the latter operator yields the maximum likelihood quantum phase estimation for an arbitrary input state [20]. Inserting $F_{\mathrm{HS}}(n)$ from (68) into (47) one will reobtain the harmonious states introduced and discussed in detail by Sudarshan [21], restricted again to a unit disk in the complex plane. On the other hand, substituting this nonlinearity function into (57) yields the following generalized coherent states:

$$|\widetilde{z}, F\rangle^{(-1)} = \widetilde{\mathcal{N}}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{n!} |n\rangle, \tag{69}$$

with the normalization constant

$$\widetilde{\mathcal{N}}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(n!)^2} = I_0(2\sqrt{|z|^2}), \tag{70}$$

where in the last expression $I_0(x)$ is the modified Bessel function of the first kind. The domain of this set of coherent states is the whole complex plane. The resolution of the identity in this case is satisfied with the choice of a density function which may be determined as $\widetilde{\eta}(x) = 2K_0(2\sqrt{x})$. So the associated weight function will be $\widetilde{\mathcal{W}}(x) = 2K_0(2\sqrt{x})I_0(2\sqrt{x})$, where I_0 and K_0 are the modified Bessel functions of the first and third kind, respectively [13].

We refrain from graphical representations, since the numerical results with the nonlinearity function (68) are closely to that of the hydrogen-like states which have been illustrated in subsection 4.1. This fact may be expected, because the two nonlinearity functions in (65) and (68) are not far from each other especially for large n.

4.3. Gilmore–Perelomov representation of the $SU(1,1)$ group

As a final example we are interested in the Gilmore–Perelomov (GP) coherent state of $SU(1,1)$ group whose number state representation read as [22]

$$|z, \kappa\rangle_{GP}^{su(1,1)} = N(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2\kappa)}{n!}} z^n |n\rangle, \tag{71}$$

where $N(|z|^2)$ is the normalization constant which can be written in closed form as follows:

$$N(|z|^2) = (1 - |z|^2)^{-2\kappa} \Gamma(2\kappa). \tag{72}$$

Note that the expansion in (71) the label κ takes the discrete values $1/2, 1, 3/2, 2, 5/2, \ldots$. The nonlinearity function in this case is determined as [9]

$$F(n, \kappa) = \frac{1}{\sqrt{n + 2\kappa - 1}}. \tag{73}$$

According to equation (47) the corresponding coherent state associated with the inverse F-deformed annihilation-like operator using $F(n,\kappa)$ in (73) takes the following decomposition in number states:

$$|z, F\rangle_{GP}^{(-1)} = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \sqrt{\frac{n!}{\Gamma(n + 2\kappa - 1)}} z^n |n\rangle, \tag{74}$$

with the normalization constant

$$\mathcal{N}(|z|^2) = \frac{{}_2F_1(\{1,1\}; \{2\kappa\}; |z|^2)}{\Gamma(2\kappa)}, \tag{75}$$

where ${}_pF_q(\overrightarrow{a}; \overrightarrow{b}; x) = {}_pF_q(\{a_1, \ldots, a_p\}; \{b_1, \ldots, b_q\}; x)$ is the generalized hypergeometric function. These states can be defined in the unit disk. Similarly, the explicit expansion of the

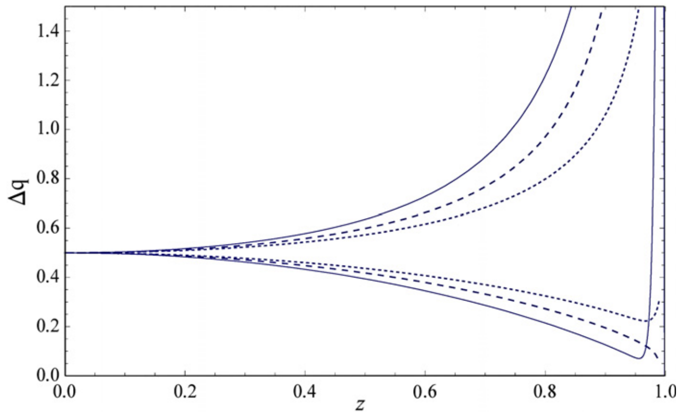

Figure 5. The uncertainties in x and p versus $z \in \mathbb{R}$ for the GP representations of $SU(1,1)$ group according to equation (47) for different values of κ; $\kappa = 0.5$ (solid lines), $\kappa = 1$ (dashed lines) and $\kappa = 1.5$ (dotted lines). Squeezing in p is observed in all cases.

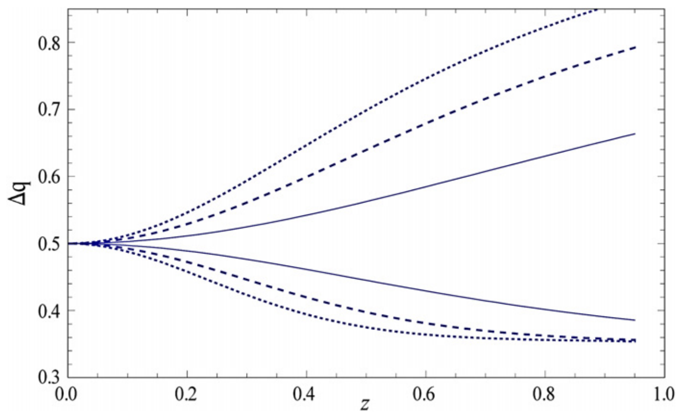

Figure 6. The same as figure 5 except that equation (57) has been considered. Squeezing in x is observed in all cases.

dual family of (74) can be obtained easily by inserting $F(n,\kappa)$ from (73) into the structural equation (57) which has the whole complex plane domain. To investigate the resolution of the identity associated with the states which have been constructed according to the structural equation (57) one gets

$$\widetilde{\eta}(x) = G^{3,0}_{1,0}\left(x \,\middle|\, \begin{matrix} . & 2\kappa - 1 \\ 0, 0, 0 & . \end{matrix}\right), \tag{76}$$

where again we have used (66). Therefore, the corresponding weight function which satisfies the resolution of the identity in this case can be evaluated as $\widetilde{\mathcal{W}}(x) = \Gamma(2\kappa)\widetilde{\eta}(x)\,{}_1F_1(2\kappa; 1; x)$, where ${}_1F_1$ is the Kummer confluent hypergeometric function and $\widetilde{\eta}(x)$ has been defined in equation (76). Now, we discuss the numerical results of the $SU(1,1)$ group in figures 5–10. In figure 5, the uncertainties in x and p have been shown with respect to $z \in \mathbb{R}$ for different values of κ according to the construction of states using (47). The squeezing in p-quadrature is visible for all values of $z < 1$, irrespective of κ values. It is seen that the maximal squeezing occurred for $\kappa = \frac{1}{2}$. Figure 6 is the same as figure 5 where (57) has been

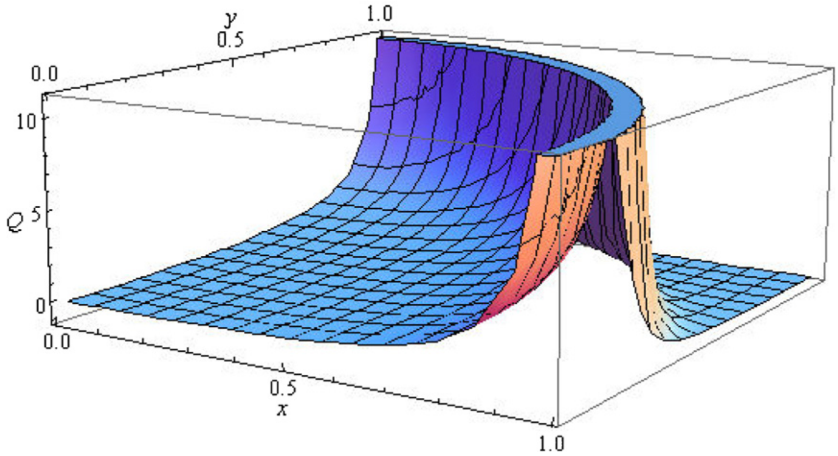

Figure 7. The Mandel's Q-parameter as a function of $z \in \mathbb{C}$ for GP representation of $SU(1, 1)$ group, according to equation (47) (κ is set equal to $1/2$).

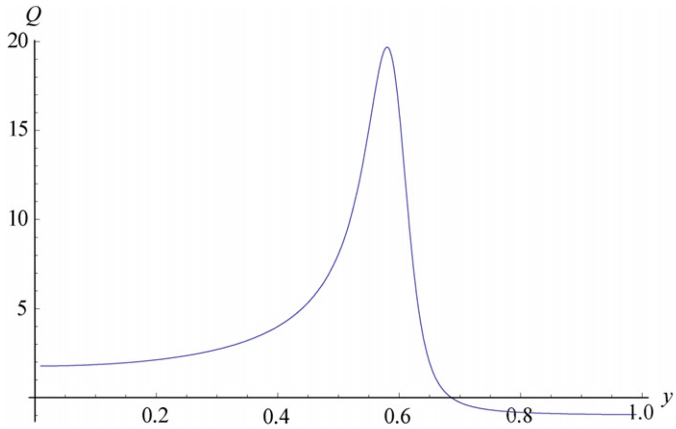

Figure 8. The two-dimensional Mandel's Q-parameter of figure 7 as a function of y (when $x = 0.8$) for GP representation of the $SU(1, 1)$ group, according to equation (47) (κ is set equal to $1/2$).

considered. In this case squeezing has been shown in x-quadrature for all values of κ, when $z \in \mathbb{R}$. The three-dimensional graphic representation of Mandel's Q-parameter for the state corresponding to (47) is plotted in figure 7. The sub-Poissonian statistics in a finite range of values of z has been shown (when both the real and imaginary parts of z are near 1, the upper bound of z). Figure 8 is a typical two-dimensional plot of figure 7 when the real part of z is fixed at a particular value, i.e. $x = 0.8$. This figure may be useful to illustrate figure 7 in detail. Figure 9 is the same as figure 7 when (57) is used. The sub-Poissonian statistics have occurred for all values of $z \in \mathbb{C}$. According to our calculations for different values of κ, the negativity of the Mandel's Q-parameter decreases (for the states of the type (57)) with increasing κ (figure 10 shows this fact when compared with figure 9). But the sub-Poissonian nature of the latter states in the complex plane preserves for all allowed values of κ. Our further calculations show that by increasing the real and imaginary parts of z the Mandel's Q-parameter fixes at a certain negative value between 0 and -1. To this end, as stated in

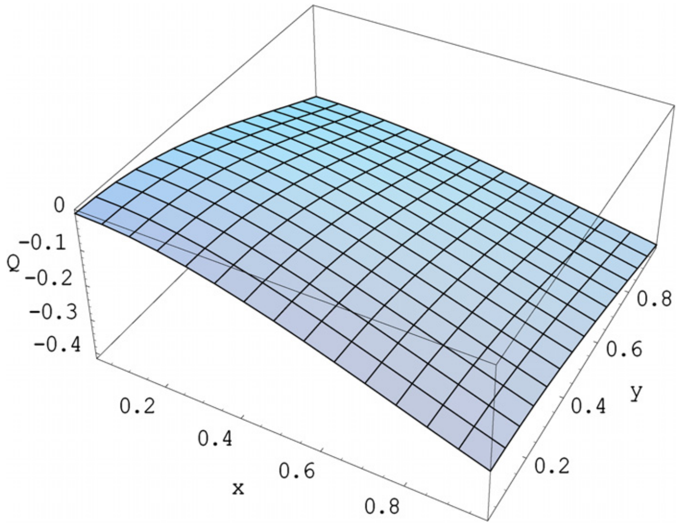

Figure 9. The same as figure 7 except that equation (57) has been considered with $\kappa = 1$.

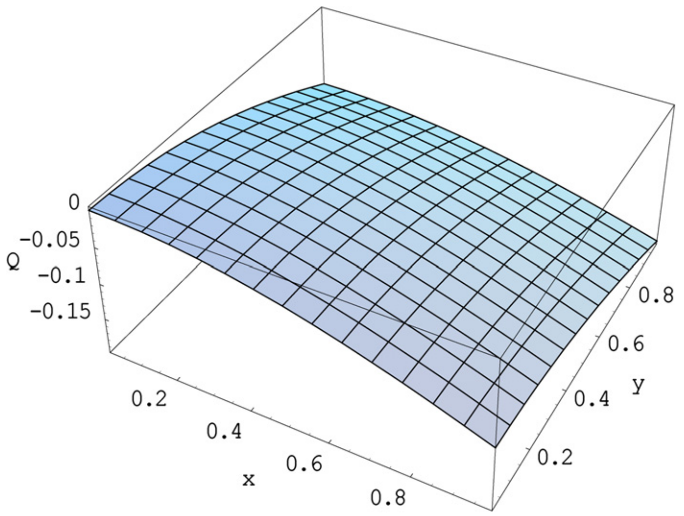

Figure 10. The same as figure 7 except that equation (57) has been considered with $\kappa = 3$.

the case of the Hydrogen-like spectrum, the latter states of $SU(1, 1)$ constructed utilizing equation (57) are also fully nonclassical states.

5. Summary and conclusion

The large number of applications of coherent states in various areas of physics motivates to enlarge them, so looking for novel definitions and new classes of states is of much interest. In this paper, based on the *'nonlinear coherent states method'*, a formalism for the construction of a coherent state associated with the *'inverse (bosonic) annihilation-like operator'* has been introduced. Although the latter was ill-defined, their dual family has been obtained in a proper fashion. Generalizing the concept, the *'inverse nonlinear (F-deformed) ladder operators'* corresponding to the deformed rising and lowering operators involved in the nonlinear coherent states of quantum optics, *'the associated nonlinear (F-deformed) coherent states'* have been introduced. The presented formalism provides a framework that by virtue of the generalized coherent states having been previously introduced (with known nonlinearity functions or corresponding to any exactly solvable potential with discrete spectrum) [23, 24], it will be possible to construct new classes of generalized coherent states associated with generalized inverse (F-deformed annihilation-like) operators. So, a large set of generalized coherent states in addition to their dual families can be constructed in the field of quantum optics. We hope that the introduced states may find their useful applications in different physical situations, both theoretically and experimentally.

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